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Quasiclassical generalized Weierstrass representation and dispersionless DS equation

B G Konopelchenko

Dipartimento di Fisica, Universita del Salento and INFN, Sezione di Lecce, 73100 Lecce, Italy

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Abstract
Quasiclassical generalized Weierstrass representation (GWR) for highly corrugated surfaces in $\mathbb{R}^4$ with a slow modulation is proposed. Integrable deformations of such surfaces are described by the dispersionless Davey–Stewartson (DS) hierarchy. Quasiclassical GWRs for other four-dimensional spaces and the dispersionless DS system are discussed too.

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1. Introduction

Classical Weierstrass representation is the basic analytic tool used to study and analyze minimal surfaces both in mathematics and in applications (see, e.g., [1–3]). Its generalizations to generic surfaces conformally immersed into the three- and four-dimensional spaces have been proposed recently in [4–8]. These generalized Weierstrass representations (GWRs) were based on the two-dimensional Dirac equation and they allow us to construct any analytic surface in R^4 and R^3. The hierarchies of the Davey–Stewartson (DS) and modified Veselov–Novikov (mVN) equations generate the integrable deformations of surfaces in R^4 and R^3, respectively, [4, 6]. GWRs provide us with an effective analytic tool to study various problems both for generic surfaces and special classes of surfaces. GWRs are also quite useful in numerous applications in applied mathematics, string theory, membrane theory and other fields of physics (see, e.g., [9–17]).

Most of the papers on this subject and results obtained have been concerned with a smooth case. On the other hand, irregular, corrugated surfaces also have attracted interest in various fields, from applied physics and technology to pure mathematics (see, e.g., [17–25]).

In the present paper, we propose a Weierstrass-type representation for highly corrugated surfaces with a slow modulation in the four- and three-dimensional Euclidean spaces. It is the quasiclassical limit of the generalized Weierstrass representation for surfaces in R^4 and $\mathbb{R}^3$ introduced in [4–6]. The quasiclassical GWR is based on the quasiclassical limit of the Dirac equation. It allows us to construct surfaces in R^4 and $\mathbb{R}^3$ with highly oscillating (corrugated) profiles and slow modulations of these oscillations characterized by a small parameter $\varepsilon = \frac{l}{L}$, where l and L are typical scales of oscillations and modulations, respectively. In the lowest order in ε the coordinates X^j $(j = 1, 2, 3, 4)$ of such surfaces in R^4 are of the form

$$\begin{aligned} X^1 + \mathrm{i}X^2 &= A(\varepsilon z, \varepsilon\bar{z}) \exp\left[\mathrm{i}\frac{S_{12}(\varepsilon z, \varepsilon\bar{z})}{\varepsilon}\right], \\ X^3 + \mathrm{i}X^4 &= B(\varepsilon z, \varepsilon\bar{z}) \exp\left[\mathrm{i}\frac{S_{34}(\varepsilon z, \varepsilon\bar{z})}{\varepsilon}\right], \end{aligned} \tag{1.1}$$

where z and $\bar{z}$ are the conformal coordinates on a surface, A and B are some smooth functions, and S_{12} and S_{34} are related to solutions of the eikonal-type equation. The corresponding metric and mean curvature are finite functions of the slow variables εz, $\varepsilon\bar{z}$ while the Gaussian curvature is of the order ε^2.

Integrable deformations of such corrugated surfaces in R^4 and R^3 are induced by the hierarchy of dispersionless DS-II (dDS) equations and dispersionless mNV (dmNV) equations. These deformations preserve the quasiclassical limit of the Willmore functional (Canham–Helfrich bending energy for membranes or the Polyakov extrinsic action for strings). The dispersionless limit of the generic DS system is considered too.

2. Generalized Weierstrass representation for surfaces in R^4

The generalized Weierstrass representation (GWR) for the surface in R^4 proposed in [6] is based on the linear systems (two-dimensional Dirac equations)

$$\Psi_{1z} = p\Phi_1, \qquad \Phi_{1\bar{z}} = -\bar{p}\Psi_1, \tag{2.1}$$

$$\Psi_{2z} = \bar{p}\Phi_2, \qquad \Phi_{2\bar{z}} = -p\Psi_2, \tag{2.2}$$

where Ψ_k and $\Phi_k (k = 1, 2)$ are complex-valued functions of $z, \bar{z} \in \mathbb{C}$ (bar denotes a complex conjugation) and $p(z, \bar{z})$ is a complex-valued function. One then defines four real-valued functions $X^j(z, \bar{z})$, $j = 1, 2, 3, 4$ by the formulae

$$X^1 + iX^2 = \int_\Gamma (-\Phi_1\Phi_2\,\mathrm{d}z' + \Psi_1\Psi_2\,\mathrm{d}\bar{z}'), \tag{2.3}$$

$$X^3 + iX^4 = -\int_\Gamma (\Phi_1\bar{\Psi}_2\,\mathrm{d}z' + \Psi_1\bar{\Phi}_2\,\mathrm{d}\bar{z}'), \tag{2.4}$$

where Γ is an arbitrary contour in $\mathbb{C}$.

Proposition 1 [6]. *For any function $p(z, \bar{z})$ and any solutions (Ψ_k, Φ_k) of the system (2.1)–(2.2), the formulae (2.3)–(2.4) define a conformal immersion of a surface into $\mathbb{R}^4$ with the induced metric*

$$\mathrm{d}s^2 = u_1u_2\,\mathrm{d}z\,\mathrm{d}\bar{z}, \tag{2.5}$$

the Gaussian curvature

$$K = -\frac{4}{u_1u_2}(\log u_1u_2)_{z\bar{z}}, \tag{2.6}$$

squared mean curvature vector

$$\mathbf{H}^2 = 2\frac{|p|^2}{u_1u_2} \tag{2.7}$$

and the Willmore functional given by

$$W \overset{def}{=} \iint_G \mathbf{H}^2[\mathrm{d}s] = 4\iint_G |p|^2 \,\mathrm{d}x\,\mathrm{d}y \tag{2.8}$$

where $u_k = |\Psi_k|^2 + |\Phi_k|^2$ *and* $z = x + \mathrm{i}y$. *Moreover, any regular surface in* $\mathbb{R}^4$ *can be constructed via the GWR (2.1)–(2.4) [6, 15].*

In the particular case $p = \overline{p}, \Psi_2 = \pm\Psi_1, \Phi_2 = \pm\Phi_1$ one has $X^4_z = X^4_{\bar{z}} = 0$ and the formulae (2.1)–(2.4) define surface in R^3 [4, 5].

Integrable dynamics of surfaces constructed via the GWR (2.1)–(2.4) is induced by the integrable evolutions of the potential $p(z, \bar{z}, t)$ and the functions Ψ_k, Φ_k with respect to the deformation parameters t_l. They are given by the DS-II hierarchy [6, 7]. The simplest example is the DS-II equation

$$\begin{aligned} &\mathrm{i}p_{t_2} + p_{zz} + p_{\bar{z}\bar{z}} + (\omega_1 + \omega_2)p = 0, \\ &\omega_{1z} = 2\,|p|^2_{\bar{z}}, \qquad \omega_{2\bar{z}} = 2\,|p|^2_z, \end{aligned} \tag{2.9}$$

for which

$$\begin{aligned} &\mathrm{i}\Psi_{1t_2} + \Psi_{1\bar{z}\bar{z}} + \omega_1\Psi_1 + p_z\Phi_1 - p\Phi_{1z} = 0 \\ &\mathrm{i}\Phi_{1t_2} + \overline{p}_{\bar{z}}\Psi_1 - \overline{p}\Psi_{1\bar{z}} - \Phi_{1zz} - \omega_2\Phi_1 = 0 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &-\mathrm{i}\Psi_{2t_2} + \Psi_{2\bar{z}\bar{z}} + \omega_1\Psi_2 + \overline{p}_z\Phi_2 - \overline{p}\Phi_{2z} = 0, \\ &-\mathrm{i}\Phi_{2t_2} + p_{\bar{z}}\Psi_2 - p\Psi_{2\bar{z}} - \Phi_{2zz} - \omega_2\Phi_2 = 0. \end{aligned} \tag{2.11}$$

In the reduction to the three-dimensional case the constraint $p = \overline{p}$ is compatible only with odd DS-II flows (times t_{2l+1}) and the DS hierarchy is reduced to the mVN hierarchy. The lowest member of this hierarchy is given by the mVN equation

$$\begin{aligned} &p_t + p_{zzz} + p_{\bar{z}\bar{z}\bar{z}} + 3\omega p_z + 3\overline{\omega}p_{\bar{z}} + \tfrac{3}{2}p\omega_z + \tfrac{3}{2}p\overline{\omega}_{\bar{z}} = 0, \\ &\omega_{\bar{z}} = (p^2)_z. \end{aligned} \tag{2.12}$$

The DS equation (2.8) and the whole DS hierarchy are amenable to the inverse spectral transform method (see, e.g., [26, 27]) and they have a number of remarkable properties typical for integrable (2+1)-dimensional equations. Integrable dynamics of surfaces in $\mathbb{R}^4$ inherits all these properties [6, 7]. One of the remarkable features of such dynamics is that the Willmore functional W (2.7) remains invariant ($W_t = 0$) [6, 7]. In virtue of the linearity of the basic problem (2.1) the GWR is quite a useful tool to study the various problems in physics and mathematics (see, e.g., [5–17]).

3. Quasiclassical Weierstrass representation

In this paper, we shall consider a class of surfaces in $\mathbb{R}^4$ which can be characterized by two scales l and L such that the parameter $\varepsilon = \frac{l}{L} \ll 1$. A simple example of such a surface is provided by the profile of a slowly modulated wavetrain for which l is a typical wavelength and L is a typical length of modulation. Theory of such highly oscillating waves with slow modulations is well developed (see, e.g., [28, 29]). Following the ideas of this Whitham (or nonlinear WKB) theory we will study surfaces in $\mathbb{R}^4$ for which the coordinates X^1, X^2, X^3, X^4 have the form

$$X^i(z, \bar{z}) = \sum_{n=0}^{\infty} \varepsilon^n F^i_n\left(\frac{\vec{S}(\varepsilon z, \varepsilon\bar{z})}{\varepsilon}, \varepsilon z, \varepsilon\bar{z}\right), \qquad i = 1, 2, 3, 4, \tag{3.1}$$

where $\overrightarrow{S} = (S^1, S^2, S^3, S^4)$ and F_n^i are smooth functions of slow variables $\xi = \varepsilon z, \overline{\xi} = \varepsilon \overline{z}$ and the small parameter ε is defined above. The arguments $\frac{S^i}{\varepsilon}$ in F_n^i describe a fast variation of a surface while the rest of arguments correspond to slow modulations.

There are different ways to specify functions F_n^i. Here we will consider one of them induced by the similar quasiclassical (WKB) limit of the GWR (2.1)–(2.4).

Thus, we begin with the quasiclassical limit of the Dirac equations (2.1)–(2.2). Following the discussion of the one-dimensional case by Zakharov [30] (dispersionless limit of the nonlinear Schrödinger equation), we take

$$p = \exp\left(\frac{\mathrm{i}(S_1(\varepsilon z, \varepsilon\overline{z}) - \widetilde{S}_1(\varepsilon z, \varepsilon\overline{z}))}{\varepsilon}\right)\sum_{n=0}^{\infty}\varepsilon^n p_n(\varepsilon z, \varepsilon\overline{z}), \tag{3.2}$$

$$\Psi_k = \exp\left(\frac{\mathrm{i}S_k(\varepsilon z, \varepsilon\overline{z})}{\varepsilon}\right)\sum_{n=0}^{\infty}\varepsilon^n \Psi_{kn}(\varepsilon z, \varepsilon\overline{z}), \tag{3.3}$$

$$\Phi_k = \exp\left(\frac{\mathrm{i}\widetilde{S}_k(\varepsilon z, \varepsilon\overline{z})}{\varepsilon}\right)\sum_{n=0}^{\infty}\varepsilon^n \Phi_{kn}(\varepsilon z, \varepsilon\overline{z}), \tag{3.4}$$

where $S_k, \widetilde{S}_k, \Psi_{kn}, \Phi_{kn}, k = 1, 2$ are smooth functions of slow variables $\xi = \varepsilon z, \overline{\xi} = \varepsilon\overline{z}, \overline{S}_k = S_k, \overline{\widetilde{S}}_k = \widetilde{S}_k$ and $S_1 + S_2 = \widetilde{S}_1 + \widetilde{S}_2$. The properties of the asymptotic expansions of the type (3.1)–(3.4) are quite well studied [28, 29]. Here we restrict ourselves to the lowest order terms. In this order one has

$$\Psi_1 = \Psi_{10}\exp\left(\frac{\mathrm{i}S_1}{\varepsilon}\right), \qquad \Phi_1 = \Phi_{10}\exp\left(\frac{\mathrm{i}\widetilde{S}_1}{\varepsilon}\right), \tag{3.5}$$

$$\Psi_2 = \Psi_{20}\exp\left(\frac{\mathrm{i}S_2}{\varepsilon}\right), \qquad \Phi_2 = \Phi_{20}\exp\left(\frac{\mathrm{i}\widetilde{S}_2}{\varepsilon}\right), \tag{3.6}$$

$$p = p_0\exp\left(\frac{\mathrm{i}(S_1 - \widetilde{S}_1)}{\varepsilon}\right). \tag{3.7}$$

Substituting these expressions into (2.1)–(2.2), one in zero order in ε gets the algebraic equations

$$\begin{pmatrix}\mathrm{i}S_{1\xi} & -p_0\\ \overline{p}_0 & \mathrm{i}\widetilde{S}_{1\overline{\xi}}\end{pmatrix}\begin{pmatrix}\Psi_{10}\\ \Phi_{10}\end{pmatrix} = 0, \qquad \begin{pmatrix}\mathrm{i}S_{2\xi} & -\overline{p}_0\\ p_0 & \mathrm{i}\widetilde{S}_{2\overline{\xi}}\end{pmatrix}\begin{pmatrix}\Psi_{20}\\ \Phi_{20}\end{pmatrix} = 0. \tag{3.8}$$

The existence of nontrivial solutions for these systems imply that p_0 and $S_k, \widetilde{S}_k$ should obey the equations

$$\det\begin{pmatrix}\mathrm{i}S_{1\xi}, -p_0\\ \overline{p}_0, \mathrm{i}\widetilde{S}_{1\overline{\xi}}\end{pmatrix} = -S_{1\xi}\widetilde{S}_{1\overline{\xi}} + |p_0|^2 = 0, \tag{3.9}$$

$$\det\begin{pmatrix}\mathrm{i}S_{2\xi}, -\overline{p}_0\\ p_0, \mathrm{i}\widetilde{S}_{2\overline{\xi}}\end{pmatrix} = -S_{2\xi}\widetilde{S}_{2\overline{\xi}} + |p_0|^2 = 0. \tag{3.10}$$

Furthermore, using the differential form of (2.3)–(2.4), namely equations

$$\begin{aligned}(X^1 + \mathrm{i}X^2)_z &= -\Phi_1\Phi_2, \qquad & (X^1 + \mathrm{i}X^2)_{\overline{z}} &= \Psi_1\Psi_2,\\ (X^3 + \mathrm{i}X^4)_z &= \Phi_1\overline{\Psi}_2, \qquad & (X^3 + \mathrm{i}X^4)_{\overline{z}} &= \Psi_1\overline{\Phi}_2,\end{aligned} \tag{3.11}$$

one concludes that in the lowest order in ε one has

$$X^1 + iX^2 = \left(X_0^1 + iX_0^2\right)\exp\left(\frac{iS_{12}}{\varepsilon}\right), \tag{3.12}$$

$$X^3 + iX^4 = \left(X_0^3 + iX_0^4\right)\exp\left(\frac{iS_{34}}{\varepsilon}\right), \tag{3.13}$$

where

$$S_{12} = S_1 + S_2 = \widetilde{S}_1 + \widetilde{S}_2 \tag{3.14}$$

$$S_{34} = S_1 - \widetilde{S}_2 = \widetilde{S}_1 - S_2 \tag{3.15}$$

and

$$X_0^1 + iX_0^2 = -i\frac{\Psi_{10}\Psi_{20}}{(S_1 + S_2)_\xi} = i\frac{\Phi_{10}\Phi_{20}}{(S_1 + S_2)_{\bar{\xi}}}, \tag{3.16}$$

$$X_0^3 + iX_0^4 = -i\frac{\Phi_{10}\overline{\Psi}_{20}}{(S_1 - \widetilde{S}_2)_\xi} = -i\frac{\Psi_{10}\overline{\Phi}_{20}}{(S_1 - \widetilde{S}_2)_{\bar{\xi}}}. \tag{3.17}$$

The last two expressions in the formulae (3.16) and (3.17) are equal to each other due to equations (3.9)–(3.10) and differential cosequences of the constraint $S_1 + S_2 = \widetilde{S}_1 + \widetilde{S}_2$.

Proposition 2. *The formulae (3.12)–(3.15) and (3.8)–(3.10), (3.16), (3.17) define a conformal immersion of highly corrugated (oscillating) surface with the slow modulation into R^4. The metric of a surface is given by*

$$ds_0^2 = u_{10}u_{20}\,dz\,d\bar{z} = \frac{1}{\varepsilon^2}u_{10}u_{20}\,d\xi\,d\bar{\xi}, \tag{3.18}$$

Gaussian curvature

$$K_0 = -\varepsilon^2\frac{2}{u_{10}u_{20}}\left(\log\left(u_{10}u_{20}\right)\right)_{\xi\bar{\xi}}, \tag{3.19}$$

squared mean curvature

$$\mathbf{H}_0^2 = 4\frac{|p_0|^2}{u_{10}u_{20}} \tag{3.20}$$

and Willmore functional

$$W_0 = 4\int\!\!\int_G |p_0|^2\,dx\,dy = \frac{1}{\varepsilon^2}4\int\!\!\int_{G_\varepsilon} |p_0|^2\,d\xi_1\,d\xi_2, \tag{3.21}$$

where $u_{k0}(\xi,\bar{\xi}) = |\Psi_{k0}|^2 + |\Phi_{k0}|^2$, $k = 1, 2$, $\xi = \xi_1 + d\xi_2$ and G_ε is the rescaled domain G $(\xi_1 = \varepsilon x, \xi_2 = \varepsilon y)$.

We shall refer to these formulae as the quasiclassical GWR and the corresponding surfaces as quasiclassical surfaces. For such surfaces the metric is conformal to a smooth one, Gaussian curvature is small (of the order ε^2) while the squared mean curvature is finite and smooth. We emphasize that this quasiclassical GWR corresponds to the lowest order terms in the expansions (3.1)–(3.4). Higher order corrections, their properties and geometrical interpretation will be considered elsewhere.

In the three-dimensional case for which $p = \overline{p}, \Psi_1 = \Psi_2, \Phi_1 = \Phi_2$ one has $S_1 = \widetilde{S}_1 = S_2 = \widetilde{S}_2$. Using the formulae (3.11) in this case one concludes that the quasiclassical GWR in R^3 is given by the formulae

$$X^1 + \mathrm{i}X^2 = -\mathrm{i}\frac{\Psi_{10}^2}{2S_\xi}\exp\left(\frac{2\mathrm{i}S_1}{\varepsilon}\right) \qquad X^3 = \frac{1}{\varepsilon}B(\xi,\overline{\xi}) \qquad B_\xi = \Phi_{10}\overline{\Psi}_{10} \tag{3.22}$$

and

$$S_{1\xi}S_{1\overline{\xi}} = p_0^2 \tag{3.23}$$

The metric, Gaussian curvature, mean curvature and Willmore functional are given by formulae (3.18)–(3.21) with $u_{10} = u_{20} = 2\,|\Psi_{10}|^2$. For more details on the quasiclassical GWR in R^3 see [31].

4. Integrable deformations via the dispersionless DS-II hierarchy

Deformations of quasiclassical surfaces described above are given by the dispersionless limit of the DS-II hierarchy. To get this limit one, as usual (at the (1+1)-dimensional case, see, e.g., [30]), assumes that the dependence of all quantities on t is a slow one, i.e. $p_0 = p_0(\varepsilon z, \varepsilon\overline{z}, \varepsilon t)$, $S_k = S_k(\varepsilon z, \varepsilon\overline{z}, \varepsilon t)$ and so on. At the first order in ε equation (2.8) gives ($\tau = \varepsilon t_2$)

$$\begin{aligned} &S_\tau + S_\xi^2 + S_{\overline{\xi}}^2 - (\omega_{10} + \omega_{20}) = 0,\\ &\omega_{10\xi} = 2\,|p_0|^2_{\overline{\xi}}, \qquad \omega_{20\overline{\xi}} = 2\,|p_0|^2_{\xi}, \end{aligned} \tag{4.1}$$

while from the system (2.9) one gets

$$\begin{pmatrix} i(S_{1\tau} + S_{1\overline{\xi}}^2 - \omega_{10}), -p_0(2\widetilde{S}_{1\xi} - S_{1\xi}) \\ \overline{p}_0(-\widetilde{S}_{1\overline{\xi}} + 2S_{1\overline{\xi}}), -i(\widetilde{S}_{1\tau} - \widetilde{S}_{1\xi}^2 + \omega_{20}) \end{pmatrix}\begin{pmatrix}\Psi_{10}\\ \Phi_{10}\end{pmatrix} = 0. \tag{4.2}$$

The system (2.10) give rises to a systems similar to this. The existence of nontrivial solutions for the system (3.8) and (4.2) implies the following independent constraints:

$$S_{1\xi}\widetilde{S}_{1\overline{\xi}} - |p_0|^2 = 0, \tag{4.3}$$

$$S_{1\tau} + S_{1\xi}^2 + S_{1\overline{\xi}}^2 - 2S_{1\xi}\widetilde{S}_{1\xi} - \omega_{10} = 0, \tag{4.4}$$

$$\widetilde{S}_{1\tau} - \widetilde{S}_{1\xi}^2 - \widetilde{S}_{1\overline{\xi}}^2 + 2S_{1\overline{\xi}}\widetilde{S}_{1\overline{\xi}} + \omega_{20} = 0. \tag{4.5}$$

This system and similar system for S_2 and $\widetilde{S}_2$ define deformation of a surface induced by the quasiclassical GWR.

The compatibility condition for the system (4.3)–(4.5) implies the following equation for $U = |p_0|^2$:

$$U_\tau + 2(US_\xi)_\xi + 2(US_{\overline{\xi}})_{\overline{\xi}} = 0. \tag{4.6}$$

The difference of equations (4.4) and (4.5) coincides with equation (4.1) which can be written also as

$$S_\tau + S_\xi^2 + S_{\overline{\xi}}^2 + V = 0, \qquad V_{\xi\overline{\xi}} + 2U_{\xi\xi} + 2U_{\overline{\xi}\overline{\xi}} = 0, \tag{4.7}$$

where $V = -(\omega_{10} + \omega_{20})$. We will refer to the system (4.6) and (4.7) as the dispersionless DS-II equation. It is the (2+1)-dimensional integrable extension of the dispersionless nonlinear Schrödinger equation studied in [30].

Equation (4.6) implies that for surfaces with rapidly decreasing U as $|\xi| \to 0$ or for compact surfaces

$$W_{0\tau} = \frac{\partial}{\partial\tau}\left(\frac{1}{\varepsilon^2}\int\int_{G_\varepsilon} U\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}\right) = 0. \tag{4.8}$$

Thus, deformation of quasiclassical surfaces via the dispersionless DS-II equation preserves the value of the Willmore functional (3.21).

Similarly, the dispersionless DS-II hierarchy which can be constructed in a same manner defines integrable deformations of quasiclassical surfaces generated by quasiclassical GWR. The Willmore functional (3.21) remains invariant under all these deformations.

For surfaces in R^3 generated by quasiclassical GWR (3.22) and (3.23) integrable deformations are induced by the dispersionless mVN hierarchy the lowest member of which is given by the dispersionless limit of equation (2.11), i.e. by the equation

$$p_{0\tau} + 3\omega_0 p_{0\xi} + 3\overline{\omega}_0 p_{0\bar{\xi}} + \tfrac{3}{2} p_0\omega_{0\xi} + \tfrac{3}{2} p_0\overline{\omega}_{0\bar{\xi}} = 0, \qquad \omega_{0\bar{\xi}} = \left(p_0^2\right)_\xi. \tag{4.9}$$

For more details on this case see [31].

5. GWRs in other four-dimensional spaces and dispersionless DS system

GWRs for surfaces and time-like surfaces in $R^{2,2}$ and Minkovsky space $R^{1,3}$ are rather similar to that in R^4 [7]. They are based on the general Dirac system

$$\Psi_x = p\Phi, \qquad \Phi_y = q\Psi, \tag{5.1}$$

where all variables are complex valued and in concrete cases x and y are real or complex conjugated and special constraints of the type $q = \overline{p}$ or $p = \overline{p}, q = \overline{q}$ are imposed. Deformations are defined by the well-known generic DS hierarchy (two-component Kadomtsev–Petviashvili (KP) hierarchy). Its lowest member is given by the DS system (see, e.g., [26, 27])

$$\begin{aligned} \alpha p_t &= p_{xx} + p_{yy} + Vp,\\ \alpha q_t &= -q_{xx} - q_{yy} - Vq,\\ V_{xy} &+ 2\,(pq)_{xx} + 2\,(pq)_{yy} = 0, \end{aligned} \tag{5.2}$$

where α is a parameter. This system is equivalent to the compatibility condition for the system (5.1) and the system

$$\begin{aligned} \alpha\Psi_t &= \Psi_{yy} + \omega_1\Psi + p_x\Phi - p\Phi_x,\\ \alpha\Phi_t &= -q_y\Psi + q\Psi_y - \Phi_{xx} - \omega_2\Phi, \end{aligned} \tag{5.3}$$

where $\omega_{1x} = -2(pq)_y, \omega_{2y} = -2(pq)_x, V = \omega_1 + \omega_2$.

The quasiclassical version of these formulae and the corresponding quasiclassical GWRs are readily obtained similar to the previous sections. Here, we will consider only the dispersionless limit of the system (5.2) since it is of interest also of its own.

First, we introduce the slow variables $\xi = \varepsilon x, \eta = \varepsilon y, \tau = \varepsilon t$ where ε is a small parameter and assume that in the lowest order in ε

$$\begin{aligned} \Psi &= \Psi_0 \exp\left(\frac{S_1}{\varepsilon}\right), & \Phi &= \Phi_0 \exp\left(\frac{S_2}{\varepsilon}\right),\\ p &= p_0 \exp\left(\frac{S_1 - S_2}{\varepsilon}\right), & q &= q_0 \exp\left(\frac{S_2 - S_1}{\varepsilon}\right), \end{aligned} \tag{5.4}$$

where Ψ_0, Φ_0, p_0, q_0 are smooth functions of slow variables. Substituting these expressions into the linear problems (5.1) and (5.3), one, in the zero order in ε, gets the system of four equations

$$S_{1\xi}\Psi_0 - p_0\Phi_0 = 0, \qquad q_0\Psi_0 - S_{2\eta}\Phi_0 = 0, \tag{5.5}$$

$$\begin{aligned} &\left(\alpha S_{1\tau} - S_{1\eta}^2 - \omega_{10}\right)\Psi_0 - (S_{1\xi} - 2S_{2\xi})p_0\Phi_0 = 0,\\ &\left(S_{2\eta} - 2S_{1\eta}\right)q_0\Psi_0 + \left(\alpha S_{2\tau} + S_{2\xi}^2 + \omega_{20}\right)\Phi_0 = 0. \end{aligned} \tag{5.6}$$

Equating to zero determinants for all subsystems of the system of equations (5.5) and (5.6), composed of any two equations from them, or simply eliminating Ψ_0 and Φ_0 from this system, one gets the following system of three independent equations:

$$\begin{aligned} &S_{1\xi}S_{2\eta} - p_0q_0 = 0,\\ &\alpha S_{1\tau} - S_{1\xi}^2 - S_{1\eta}^2 + 2S_{1\xi}S_{2\eta} - \omega_{10} = 0,\\ &\alpha S_{2\tau} + S_{2\xi}^2 + S_{2\eta}^2 - 2S_{1\eta}S_{2\eta} + \omega_{20} = 0. \end{aligned} \tag{5.7}$$

The compatibility condition for these equations is equivalent to the equation

$$\alpha U_{0\tau} - 2\nabla(U_0\nabla S) = 0, \tag{5.8}$$

where $U_0 = p_0q_0$, $S = S_1 - S_2$ and $\nabla = (\partial_\xi, \partial_\eta)$ while the difference of the second and third equations (5.7) gives

$$\alpha S_\tau - (\nabla S)^2 + V_0 = 0, \tag{5.9}$$

where

$$V_{0\xi\eta} - 2\triangle U_0 = 0 \tag{5.10}$$

and $\triangle = \partial_\xi^2 + \partial_\eta^2$. The system (5.8)–(5.10) represents the dispersionless limit of the DS system (5.2). We note that equations (5.9) and (5.10) are just the dispersionless limit of equations (5.2) while equation (5.8) is the dispersionless limit of the first conservation law $\alpha(pq)_t + (pq_x - p_xq)_x + (pq_y - p_yq)_y = 0$ for the DS system (5.2). At $\alpha = -\mathrm{i}$ and $q_0 = -\overline{p}_0$ this system coincides with the system (4.6) and (4.7).

The system (5.8)–(5.10) is quite close in the form to the classical hydrodynamical equations of shallow water for gradient flows (see, e.g., [30]). In the one-dimensional limit $\partial_\xi = \partial_\eta$ it coincides with the one-dimensional Benney system (see [30]). So the system (5.8)–(5.10) is its two-dimensional integrable generalization.

It differs from the (2+1)-dimensional extension of the Benney system, namely

$$a_\tau + (au)_\xi = 0, \qquad u_\tau + \tfrac{1}{2}(u^2)_\xi + \omega_\xi = 0, \qquad \omega_\eta + a_\xi = 0, \tag{5.11}$$

given in [32, 33] which is equivalent to the compatibility condition for the system of Hamilton–Jacobi equations

$$\chi_\eta = \frac{a}{\chi_\xi - u}, \qquad \chi_\tau = -\tfrac{1}{2}\chi_\xi^2 - \omega. \tag{5.12}$$

To compare this system with (5.7) it is instructive to rewrite (5.7) in the equivalent form introducing S and χ defined by $S_1 = S + \chi$, $S_2 = \chi$. In these variables, the first and second equations (5.7) take the form

$$\chi_\eta = -\frac{U_0}{\chi_\xi + S_\xi}, \qquad \chi_\tau = \chi_\xi^2 - \chi_\eta^2 - 2S_\eta\chi_\eta - \omega_{20}. \tag{5.13}$$

Though the only difference between (5.12) and (5.13) is in their time parts, the system (5.8)–(5.10) is symmetric in ξ and η in contrast to the system (5.11).

The dispersionless limit of the multi-component KP hierarchy has been discussed recently in [34]. But, it seems, that the system (5.8)–(5.10) was missed there.

Finally, it is worth noting that the system (5.8)–(5.10) has a natural interpretation within the classical mechanics as the system which describes the integrable deformations of the potential V_0 in the Hamilton–Jacobi equation (5.9) driven by equations (5.7) and (5.10).

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